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Fixed point theorems for weakly F-contractive and strongly F-expansive mappings

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ABSTRACT

The main purpose of this paper is to prove a collection of new fixed point theorems for so-called weakly F-contractive mappings. By analogy, we introduce also a class of strongly F-expansive mappings and we prove fixed point theorems for such mappings. We provide a few examples, which illustrate these results and, as an application, we prove an existence and uniqueness theorem for the generalized Fredholm integral equation of the second kind. Finally, in Appendix A, we apply the Mönch fixed point theorem to prove two results on the existence of approximate fixed points of some continuous mappings.

1. Introduction

It is commonly known that if X is a compact metric space and $f : X \to X$ is a weakly contractive mapping (see Section 2 for the definition), then f has a fixed point in X (see [6, (6.1), p. 17]). In 1969 Furi and Vignoli [7] extended this result to α-condensing mappings acting in a bounded complete metric space (α denotes the Kuratowski measure of noncompactness; we refer the reader to [1] for the definition and basic properties of this index). A generalization of Furi–Vignoli's theorem to weakly F-contractive mappings acting in a topological space was proved in [3].

On the other hand, in the paper [5], examining so-called KKM maps, the authors introduced a new concept of lower (upper) semi-continuous function (see Section 2 for the definition) which is more general than the classical one. We are going to use this definition to redefine weakly F-contractive mappings as well as to define strongly F-expansive mappings. Having the modified definition of weakly F-contractive mappings we reformulate the above-mentioned fixed point theorem from [3] (for convenience of the reader we recall the short proof of that result, because we will use it in the sequel), which is an initial point of our considerations. Basing on it, we are going to prove a fixed point theorem involving the

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assumptions that a certain iteration of a mapping in question is weakly F-contractive. As a corollary from this fact, we get some generalization of a fixed point theorem from [13] for Banach spaces with a quasimodulus endowed with a suitable transitive binary relation.

Next, we prove a simple variant of a fixed point theorem for strongly expansive mappings (see [6, (6.8), p. 18]). Our version of this theorem is very useful, in particular to prove the existence and uniqueness result for so-called generalized Fredholm integral equation of the second kind.

Finally, our last fixed point theorem in Section 3 (Theorem 5), which seems to be the most important result in this note, concerns the existence and uniqueness of a fixed point of so-called strongly expansive homeomorphism.

In Section 4, except of the existence result for the Fredholm equation, we provide a few examples in metric spaces, which illustrate our considerations.

Appendix A is a complement of the results on the existence of an approximate fixed point which were proved by the first author in [3]. Let us recall that if $(X, \|\cdot\|)$ is a Banach space, $f: X \to X$, then f has a fixed point if and only if the following two conditions are satisfied:

$$\inf\{\|x - f(x)\|: x \in X\} = 0, \tag{1}$$

$$\text{the functional } x \mapsto \|x - f(x)\| \text{ has a minimum point.} \tag{2}$$

Theorem 3 [3] (being an extension of Theorem 1 [2]) gives sufficient conditions under which (1) is satisfied. In particular a mapping f has to transform a closed convex and bounded subset D of a Banach space into itself.

In Appendix A we consider more general situation, that is we do not require f to map D into itself.

For completeness, let us mention that Section 2 contains basic definitions and facts which we use in the sequel.

2. Preliminaries

At the beginning of this section we recall the definition of a lower semi-continuous from above function introduced in [5].

Definition 1. Let X be a topological space. A function $f: X \to \mathbb{R}$ is said to be *lower semi-continuous from above at x_0* if $f(x_0) \leqslant \lim_{\lambda\in\Lambda} f(x_\lambda)$ for any net $(x_\lambda)_{\lambda\in\Lambda}$ convergent to x_0 such that $f(x_{\lambda_1}) \leqslant f(x_{\lambda_2})$ for $\lambda_2 \preccurlyeq \lambda_1$.

A function $f: X \to \mathbb{R}$ is said to be *lower semi-continuous from above* if it is lower semi-continuous from above at every point $x \in X$.

Remark 1. Every lower semi-continuous function is also lower semi-continuous from above, yet the converse is not true. It is enough to consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x+1, & \text{for } x \geqslant 0, \\ x, & \text{for } x < 0 \end{cases}$$

(see [5] for the details).

The next two lemmas establish some properties of lower semi-continuous from above functions. The former one is an analogue of the Weierstrass boundedness theorem, whereas the latter one deals with the superposition of a continuous function with a function lower semi-continuous from above.

Lemma 1. *(See [5].) Let X be a compact topological space and $f: X \to \mathbb{R}$ be a lower semi-continuous from above. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$.*

Lemma 2. *Let X and Y be topological spaces. If $f: X \to Y$ is a continuous function and $g: Y \to \mathbb{R}$ is a lower semi-continuous from above, then the superposition $h = g \circ f: X \to \mathbb{R}$ is also lower semi-continuous from above.*

Proof. Take $x \in X$ and let $(x_\lambda)_{\lambda\in\Lambda}$ be an arbitrary net in X converging to x such that $h(x_{\lambda_1}) \leqslant h(x_{\lambda_2})$ for $\lambda_2 \preccurlyeq \lambda_1$. Hence, putting $y_\lambda = f(x_\lambda)$ and $y = f(x)$ we have $\lim_{\lambda\in\Lambda} y_\lambda = y$ and $g(y_{\lambda_1}) \leqslant g(y_{\lambda_2})$ for $\lambda_2 \preccurlyeq \lambda_1$. Thus, by the assumptions, $g(y) \leqslant \lim_{\lambda\in\Lambda} g(y_\lambda)$, which is equivalent to $h(x) \leqslant \lim_{\lambda\in\Lambda} h(x_\lambda)$. $\square$

Remark 2. Let functions $f: X \to X$ and $F: X \times X \to \mathbb{R}$ be continuous and lower semi-continuous from above, respectively. Then from Lemma 2 it follows that the function $x \mapsto F(x, f(x))$ is lower semi-continuous from above.

Now we are going to define weakly F-contractive and strongly F-expansive mappings.

Definition 2. Let X be a topological space and let F be a real lower semi-continuous from above function defined on $X \times X$. The mapping $f : X \to X$ is said to be:

1. *weakly F-contractive*, if the condition $F(f(x), f(y)) < F(x, y)$ holds for all $x, y \in X$ such that $x \neq y$;
2. *strongly F-expansive*, if the condition $F(f(x), f(y)) > F(x, y)$ holds for all $x, y \in X$ such that $x \neq y$.

If X is a metric space with a distance d and $F(x, y) = d(x, y)$, then we call f *weakly contractive* or *strongly expansive*, respectively.

Let us recall that the item 1 of the above definition was introduced by Furi and Vignoli in [7].

To illustrate our fixed point theorems we will use two hyperconvex metrics on the plane, namely the radial metric and the 'river' metric (see [4] for more details). Let us recall the definitions.

Definition 3. The following metric:

$$d_{ri}(v_1, v_2) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |y_1| + |y_2| + |x_1 - x_2|, & \text{if } x_1 \neq x_2, \end{cases}$$

where $v_i = (x_i, y_i) \in \mathbb{R}^2$ for $i = 1, 2$, is called the *'river' metric* in $\mathbb{R}^2$.

Definition 4. The *radial metric* in $\mathbb{R}^2$ is defined as follows:

$$d_{ra}(v_1, v_2) = \begin{cases} \varrho(v_1, v_2), & \text{if } 0 = (0, 0),\ v_1, v_2 \text{ are colinear}, \\ \varrho(v_1, 0) + \varrho(0, v_2), & \text{otherwise}, \end{cases}$$

where $v_i = (x_i, y_i) \in \mathbb{R}^2$ for $i = 1, 2$ and $\varrho(\cdot,\cdot)$ denotes the Euclidean metric in $\mathbb{R}^2$.

3. Fixed point theorems

Let us begin this section with the following

Theorem 1. *Let X be a topological space, $x_0 \in X$ and let $f : X \to X$ be a continuous and weakly F-contractive mapping. If the implication*

$$V = f(V) \cup \{x_0\} \quad \Rightarrow \quad V \text{ is relatively compact}$$

holds for every countable subset V of X, then f has a unique fixed point.

Proof. Define the sequence $(y_n)_{n \in \mathbb{N}}$ by the formulae

$$y_1 = x_0, \qquad y_{n+1} = f(y_n), \quad n \in \mathbb{N}$$

and let $A = \{y_n\colon n \in \mathbb{N}\}$. Obviously $A = f(A) \cup \{x_0\}$, so in view of the assumption A is relatively compact. Define the real function $\varphi : \overline{A} \to \mathbb{R}$ by

$$\varphi(x) = F\big(x, f(x)\big), \quad x \in \overline{A}.$$

By Remark 2 the mapping φ is lower semi-continuous from above, so by Lemma 1 it has a minimum point, say $y \in \overline{A}$. Of course, $f(y) \in \overline{A}$. Indeed, in view of the inclusion $f(A) \subset A$ and continuity of f, we have $f(y) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{A}$. Suppose that $f(y) \neq y$. Since

$$\varphi\big(f(y)\big) = F\big(f(y), f^2(y)\big) < F\big(y, f(y)\big) = \varphi(y),$$

a contradiction follows. Hence $y = f(y)$ and the weakly F-contractivity implies that y is a unique fixed point of f. $\square$

One can generalize Theorem 1 in the following way.

Theorem 2. *Let X be a topological space and let $f : X \to X$ be a continuous mapping such that for some $k \in \mathbb{N}$ the kth iteration f^k is weakly F-contractive. Suppose that one of the following conditions is satisfied*:

(j) *there exists $x_0 \in X$ such that the implication*

$$V = f(V) \cup \{x_0\} \quad \Rightarrow \quad V \text{ is relatively compact}$$

holds for every countable subset $V \subset X$;

(jj) *there exists* $x_0 \in X$ *such that the implication*

$$V = f^k(V) \cup \{x_0\} \quad \Rightarrow \quad V \text{ is relatively compact}$$

holds for every countable subset $V \subset X$.

Then the function f *has a unique fixed point.*

Proof. Let us define the mapping φ as $\varphi(x) = F(x, f^k(x))$ for $x \in X$. The reasoning as in the proof of Theorem 1 leads us to the statement that f^k has the unique fixed point y. The point y is of course the unique fixed point of f. That proves part (j). By putting $g = f^k$, part (jj) follows from Theorem 1. □

Basing on Theorem 2 we can prove the following fixed point theorem for mappings in a Banach space with quasimodulus.

Theorem 3. *Let* X *be a Banach space with a transitive binary relation* $\preccurlyeq$ *such that*

(i) *the norm* $\|\cdot\|$ *in* X *is monotonic, that is, for* $x, y \in X$ *if* $x \preccurlyeq y$, *then* $\|x\| \leqslant \|y\|$.

Furthermore, let three mappings be given $f, m, A : X \to X$ *such that*

(ii) $\theta \preccurlyeq m(x)$ *and* $\|m(x)\| = \|x\|$ *for all* $x \in X$;
(iii) A *is bounded linear operator and* $\|A^k x\| < \|x\|$ *for some* $k \in \mathbb{N}$ *and all* $x \in X$ *such that* $x \neq \theta$ *with* $\theta \preccurlyeq x$;
(iv) *if* $\theta \preccurlyeq x \preccurlyeq y$, *then* $Ax \preccurlyeq Ay$;
(v) $m(f(x) - f(y)) \preccurlyeq Am(x - y)$ *for all* $x, y \in X$.

If one of the conditions in Theorem 2 *holds* (*condition* (jj) *with the same* k *as assumption* (iii) *of this theorem*) *then the function* f *has a unique fixed point.*

Proof. We will show that f satisfies the assumptions of Theorem 2. For $x, y \in X$ we have

$$m\big(f^2(x) - f^2(y)\big) \preccurlyeq Am\big(f(x) - f(y)\big) \preccurlyeq A^2 m(x - y)$$

and similarly

$$m\big(f^k(x) - f^k(y)\big) \preccurlyeq A^k m(x - y).$$

Thus

$$\begin{aligned}\big\|f^k(x) - f^k(y)\big\| &= \big\|m\big(f^k(x) - f^k(y)\big)\big\| \leqslant \big\|A^k m(x - y)\big\| \\ &< \big\|m(x - y)\big\| = \|x - y\|.\end{aligned}$$

Therefore the kth iteration f^k is weakly contractive. Moreover, because A is bounded, the function f is continuous. In view of Theorem 2 f has a unique fixed point. □

Example 1. As usual, by c_0 let us denote the space of all sequences which are convergent to zero with the sup norm. Define the operator $A : c_0 \to c_0$ by the following formula

$$Ax = \left(\xi_2, \xi_3, \frac{1}{2}\xi_4, \frac{1}{2}\xi_5, \ldots\right) \quad \text{for } x = (\xi_n)_{n \in \mathbb{N}}.$$

Then $\|Ax\| \leqslant \|x\|$ for all $x \in c_0$, but $\|A^3 x\| < \|x\|$ for all $x \in c_0$ such that $x \neq \theta$.

Now, let pass on to strongly expansive mappings. Firstly, let us modify the well-known fixed point theorem from [6, (6.8), p. 18]. We will apply this modification in the next section examining the generalized Fredholm integral equation of the second kind.

Theorem 4. *Let* (Y, d) *be a metric space,* $A \subset B \subset Y$ *with* B *complete. If* $f : A \to B$ *is a surjective mapping such that*

$$d\big(f(x), f(y)\big) \geqslant \beta\, d(x, y) \quad \text{for all } x, y \in A,$$

where β *is a constant greater than* 1, *then* f *has a unique fixed point.*

Proof. It is easy to check that f is injective. Indeed, if $f(x) = f(y)$, then $d(f(x), f(y)) = 0$, so $d(x, y) = 0$ and thus $x = y$. Let us consider the mapping $f^{-1} : B \to A \subset B$. Take $z, w \in B$. There exist $x, y \in A$ such that $z = f(x)$, $w = f(y)$. We have

$$d(z, w) = d\big(f(x), f(y)\big) \geqslant \beta d(x, y) = \beta d\big(f^{-1}(z), f^{-1}(w)\big),$$

so

$$d\big(f^{-1}(z), f^{-1}(w)\big) \leqslant \frac{1}{\beta} d(z, w), \quad \frac{1}{\beta} < 1.$$

In view of the Banach contraction principle there exists a unique fixed point of the mapping f^{-1}, say x_0. Then $x_0 = f^{-1}(x_0)$, so $x_0 = f(x_0)$ and it is a unique fixed point of f. □

The following fixed point theorem seems to be the most important result in this section.

Theorem 5. *Let X be a topological space, $A \subset B \subset X$ and $f : A \to B$ be a strongly F-expansive homeomorphism. If there exists $x_0 \in A$ such that the following implication*

$$f(C) = C \cup \big\{f(x_0)\big\} \quad \Rightarrow \quad C \text{ is relatively compact in } A \tag{3}$$

holds for every countable subset C of A, then f has a unique fixed point.

Proof. Obviously, $f^{-1} : B \to A \subset B$. For any $z, w \in B$ $(z \neq w)$, we have

$$F(z, w) = F\big(f(x), f(y)\big) > F(x, y) = F\big(f^{-1}(z), f^{-1}(w)\big),$$

where $z = f(x)$ and $w = f(y)$, so

$$F\big(f^{-1}(z), f^{-1}(w)\big) < F(z, w),$$

and therefore f^{-1} is a weakly F-contractive mapping. Let V be any countable subset of B such that

$$V = f^{-1}(V) \cup \{y_0\},$$

where $y_0 = f(x_0)$. Then $V = f(C)$ for some $C \subset A$. We have

$$f(C) = f^{-1}\big(f(C)\big) \cup \big\{f(x_0)\big\},$$

so

$$f(C) = C \cup \big\{f(x_0)\big\}.$$

By (3) the set $\overline{C}$ is compact, so the continuity of f implies that $\overline{V}$ is compact, too. By Theorem 1, f^{-1} has exactly one fixed point, say x_0; that is $f^{-1}(x_0) = x_0$ and it is a unique fixed point of f. □

4. Examples and applications

At the beginning of this section we are going to provide three examples which illustrate Theorem 5.

Example 2. Let us consider $\mathbb{R}^2$ with the 'river' metric,

$$A = \big\{(x, y) \in \mathbb{R}^2\colon\ x = y,\ x \in [0, 1]\big\},$$

and

$$B = \big\{(x, y) \in \mathbb{R}^2\colon\ x = y,\ x \in [0, 2]\big\}.$$

Let $f : A \to B$ be defined as follows: $f((x, y)) = (2x, 2y)$. Of course f satisfies (3). Indeed, let us take $x_0 = (0, 0)$ and an arbitrary countable subset C of A such that $f(C) = C \cup \{(0, 0)\}$. Note that in this case $C \subset \{(0, 0)\}$, since if $(x, y) \neq (0, 0)$ belonged to C, then $f^n((x, y)) \in C \subset A$ for every $n \in \mathbb{N}$, which is impossible. Moreover, it is very easy to establish that f is strongly F-expansive mapping, if one puts

$$F(z, w) = d_{ra}(z, w),$$

where d_{ra} is the radial metric defined on $\mathbb{R}^2$. Namely

$$d_{ra}\big(f(z), f(w)\big) = 2d_{ra}(z, w) \quad \text{for } z, w \in A$$

and $d_{ra} : (\mathbb{R}^2, d_{ri}) \times (\mathbb{R}^2, d_{ri}) \to [0, +\infty)$ is lower semi-continuous, hence, in view of Remark 1, lower semi-continuous from above, too. By Theorem 5 the mapping f has a unique fixed point.

Example 3. Let X be the space of all continuous functions $x:[-1,1]\to\mathbb{R}$ with the norm

$$\|x\|_2=\left(\int_{-1}^{1}\left|x(t)\right|^2dt\right)^{\frac{1}{2}}.$$

Define the mapping $f:A\to B$ as $f(x)=2x$ for $x\in A$, where

$$A=\left\{x\in X:\ \|x\|_2\leqslant 1\right\}$$

and

$$B=\left\{x\in X:\ \|x\|_2\leqslant 2\right\}.$$

Note that B is not complete. Obviously

$$\left\|f(x)-f(y)\right\|_2\geqslant 2\|x-y\|_2,$$

so f is strongly expansive mapping. Moreover, it is easy to see, that f satisfies (3) with $x_0=0\in X$. Since f as well as f^{-1} are continuous, we can apply Theorem 5 and claim that f has a unique fixed point.

Example 4. Let us consider $\mathbb{R}^2$ with the radial metric and let

$$A=\left\{(x,y)\in\mathbb{R}^2:\ x\in[9,10],\ \ y=20-2x\right\},$$

and

$$B=\left\{(x,y)\in\mathbb{R}^2:\ x\in[8,10],\ \ y=20-2x\right\}.$$

Let us define the mapping $f:A\to B$ as $f((x,y))=(2x-10,2y)$. It is easy to see, that f is a homeomorphism, since from the topological point of view the domains of f and f^{-1} consist of isolated points. Take $(x_i,y_i)\in A$, $i=1,2$. We may assume that $x_1<x_2$. Then

$$d_{ri}\big(f(x_1,y_1),f(x_2,y_2)\big)=2y_1+2x_2-2x_1+2y_2=2d_{ri}\big((x_1,y_1),(x_2,y_2)\big),$$

where d_{ri} denotes the 'river' metric. It can be verified that the 'river' metric $d_{ri}:(\mathbb{R}^2,d_{ra})\times(\mathbb{R}^2,d_{ra})\to[0,+\infty)$ is lower semi-continuous, hence, in view of Remark 1, lower semi-continuous from above, too. Additionally, putting $x_0=(10,0)$ in (3), one can readily check that (3) is satisfied. Hence by Theorem 5, the function f has a fixed point in A.

However, f is not strongly d_{ra}-expansive. Indeed,

$$d_{ra}\big(f(10,0),f(9,2)\big)=10+\sqrt{80}$$

and

$$d_{ra}\big((10,0),(9,2)\big)=10+\sqrt{85}.$$

It is worth noting, then, that the introduction of the function F in Theorem 5 is essential.

In the second part of this section we are going to show an application of Theorem 4 to integral equations.

Let us consider the following Fredholm integral equation of the second kind

$$f(x)+\lambda\int_a^b k(x,t)\varphi(t)\,dt=\varphi(x),\quad a\leqslant x\leqslant b,\tag{4}$$

where $\lambda\in\mathbb{R}$ and the real valued functions f as well as k are defined and square integrable in the Lebesgue sense on $[a,b]$ and $[a,b]\times[a,b]$, respectively. If the kernel k is hermitian, that is $k(x,t)=k(t,x)$ for almost all x and t (we do not need the conjugation, since we consider only real Hilbert spaces) or equivalently the integral operator $K:L_2(a,b)\to L_2(a,b)$ associated with this kernel is self-adjoint, then according to the Spectral Theorem (see [9, Th. 4.15, p. 109]) we can represent the operator K on $L_2(a,b)$ as follows

$$K\varphi=\sum_{n=1}^{\infty}\mu_n\langle\varphi,\varphi_n\rangle\varphi_n,$$

where $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of nonzero eigenvalues of K and $(\varphi_n)_{n\in\mathbb{N}}$ is a corresponding orthonormal sequence of eigenvectors. (The above sum is a finite sum, if there are only finitely many eigenvalues.) Hence, we can rewrite Eq. (4) in the form

$$f+\lambda\sum_{n=1}^{\infty}\mu_n\langle\varphi,\varphi_n\rangle\varphi_n=\varphi.\tag{5}$$

It is well known that the sequence of eigenvalues $(\mu_n)_{n\in\mathbb{N}} \in l^2$ (see [9, Th. 4.19, p. 120]), thus $\inf_{n\in\mathbb{N}} \inf_{x\in\mathbb{R}} |u_n'(x)| = 0$, where $u_n : \mathbb{R} \to \mathbb{R}$ is given by the formula $u_n(x) = \mu_n x$.

In the following example we deal with a nonlinear equation in the real Banach space $L_2(a,b)$, which is given in the form similar to (5), yet we will assume that $\inf_{n\in\mathbb{N}} \inf_{x\in\mathbb{R}} |u_n'(x)| > 0$.

Example 5. Let $\{k_n : \mathbb{R} \to \mathbb{R}\colon\ n \in \mathbb{N}\}$ be a family of functions such that for every $n \in \mathbb{N}$

(i) $k_n \in C^1(\mathbb{R})$;
(ii) $k_n(0) = 0$;

and

(iii) $\inf_{n\in\mathbb{N}} \inf_{x\in\mathbb{R}} |k_n^{'}(x)| \geqslant \kappa > 0$.

(For example, the collection

$$\Big\{\varkappa_n : \mathbb{R} \to \mathbb{R}\colon\ \varkappa_n(x) = \alpha_n x + e^x - 1,\ \text{where } \inf_{n\in\mathbb{N}} \alpha_n \geqslant \alpha > 0\Big\}$$

satisfies the above conditions.) Then, of course, for every $n \in \mathbb{N}$ the inverse function $k_n^{-1} : \mathbb{R} \to \mathbb{R}$ exists.

Let us consider the real Banach space of square integrable functions $L_2(a,b)$, for some $a, b \in \mathbb{R}$ such that $a < b$. By $(\varphi_n)_{n\in\mathbb{N}}$ we denote a complete orthonormal system of functions in this space. Take an arbitrary function $f \in L_2(a,b)$ with

$$\sum_{n=1}^{\infty} \big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big|^2 < +\infty,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product. We prove that in fact an arbitrary function $f \in L_2(a,b)$ satisfies the above condition. For every $n \geqslant 2$ we have

$$\begin{aligned}
\kappa\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| &= \kappa\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big) - k_n^{-1}\big(\langle \varphi_1, \varphi_n\rangle\big)\big| \\
&\leqslant \big|k_n'(\theta_n)\big|\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big) - k_n^{-1}\big(\langle \varphi_1, \varphi_n\rangle\big)\big| \\
&= \big|k_n \circ k_n^{-1}\big(\langle f, \varphi_n\rangle\big) - k_n \circ k_n^{-1}\big(\langle \varphi_1, \varphi_n\rangle\big)\big| \\
&= \big|\langle f, \varphi_n\rangle - \langle \varphi_1, \varphi_n\rangle\big| = \big|\langle f - \varphi_1, \varphi_n\rangle\big|,
\end{aligned}$$

where $\theta_n \in \mathbb{R}$ is taken according to the Lagrange Mean Value Theorem. Since $f - \varphi_1 \in L_2(a,b)$, the condition in question is satisfied.

Let us define sets

$$A = \left\{\varphi \in L_2(a,b)\colon\ \sum_{n=1}^{\infty} \big|k_n\big(\langle \varphi, \varphi_n\rangle\big)\big|^2 < +\infty\right\}$$

as well as

$$B = \left\{\varphi \in L_2(a,b)\colon\ \sum_{n=1}^{\infty} \big|k_n^{-1}\big(\langle \varphi + f, \varphi_n\rangle\big)\big|^2 < +\infty\right\}.$$

In general $A \neq L_2(a,b)$. Indeed, let $\xi_n = n^{-1}$ for $n \in \mathbb{N}$. Then $(\xi_n)_{n\in\mathbb{N}} \in l_2$ and of course $\varphi = \sum_{n=1}^{\infty} \xi_n \varphi_n \in L_2(a,b)$. However, if we put $k_n(x) = nx$ for $x \in \mathbb{R}$, then it is easy to check that such a family of functions satisfies all assumptions, yet $(k_n(\xi_n))_{n\in\mathbb{N}} \notin l_2$.

We are going to show now that $B = L_2(a,b)$. For that purpose let us take an arbitrary $\varphi \in L_2(a,b)$. Then for every $n \in \mathbb{N}$ we have:

$$\begin{aligned}
\kappa\big|k_n^{-1}\big(\langle \varphi + f, \varphi_n\rangle\big)\big| &\leqslant \kappa\big|k_n^{-1}\big(\langle \varphi + f, \varphi_n\rangle\big) - k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| + \kappa\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| \\
&\leqslant \big|k_n'(\theta_n)\big|\big|k_n^{-1}\big(\langle \varphi + f, \varphi_n\rangle\big) - k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| + \kappa\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| \\
&= \big|k_n \circ k_n^{-1}\big(\langle \varphi + f, \varphi_n\rangle\big) - k_n \circ k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| + \kappa\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| \\
&= \big|\langle \varphi + f, \varphi_n\rangle - \langle f, \varphi_n\rangle\big| + \kappa\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big| \\
&= \big|\langle \varphi, \varphi_n\rangle\big| + \kappa\big|k_n^{-1}\big(\langle f, \varphi_n\rangle\big)\big|,
\end{aligned}$$

where $\theta_n \in \mathbb{R}$ is taken according to the Lagrange Mean Value Theorem. Thus $\varphi \in B$ and $B = L_2(a,b)$. In particular, B is a complete set in the norm $\|\cdot\|_2$ and $A \subset B$.

For $\varphi \in A$ and $\lambda \in \mathbb{R} \setminus \{0\}$ let us consider the following equation

$$\lambda \sum_{n=1}^{\infty} k_n\big(\langle \varphi, \varphi_n \rangle\big)\varphi_n - \varphi = f. \tag{6}$$

Now, we are about to prove that the mapping $K_\lambda : A \to B$ given by the formula:

$$K_\lambda(\varphi) = \lambda \sum_{n=1}^{\infty} k_n\big(\langle \varphi, \varphi_n \rangle\big)\varphi_n - f$$

is surjective. Let us take $g \in B$ and let us define $\gamma = \sum_{n=1}^{\infty} k_n^{-1}(\lambda^{-1}\langle g + f, \varphi_n\rangle)\varphi_n$. Because

$$\big|k_n \circ k_n^{-1}\big(\lambda^{-1}\langle g + f, \varphi_n\rangle\big)\big| = \big|\lambda^{-1}\langle g + f, \varphi_n\rangle\big| \quad \text{for } n \in \mathbb{N},$$

we have $\gamma \in A$. Moreover

$$\begin{aligned} K_\lambda(\gamma) &= \lambda \sum_{n=1}^{\infty} k_n\left(\left\langle \sum_{m=1}^{\infty} k_m^{-1}\big(\lambda^{-1}\langle g + f, \varphi_m\rangle\big)\varphi_m, \varphi_n \right\rangle\right)\varphi_n - f \\ &= \lambda \sum_{n=1}^{\infty} k_n \circ k_n^{-1}\big(\lambda^{-1}\langle g + f, \varphi_n\rangle\big)\varphi_n - f \\ &= g + f - f = g. \end{aligned}$$

For arbitrary $\varphi, \psi \in A$, in view of the Parseval identity (see [12, p. 86]) and the assumption (iii) we obtain

$$\begin{aligned} \big\|K_\lambda(\varphi) - K_\lambda(\psi)\big\|_2^2 &= |\lambda|^2 \sum_{n=1}^{\infty} \big|k_n\big(\langle \varphi, \varphi_n\rangle\big) - k_n\big(\langle \psi, \varphi_n\rangle\big)\big|^2 \\ &= |\lambda|^2 \sum_{n=1}^{\infty} \big|k_n'(\theta_n)\big|^2 \big|\langle \varphi, \varphi_n\rangle - \langle \psi, \varphi_n\rangle\big|^2 \\ &\geqslant |\lambda\kappa|^2 \sum_{n=1}^{\infty} \big|\langle \varphi, \varphi_n\rangle - \langle \psi, \varphi_n\rangle\big|^2 = |\lambda\kappa|^2 \|\varphi - \psi\|_2^2 \end{aligned}$$

and therefore

$$\big\|K_\lambda(\varphi) - K_\lambda(\psi)\big\|_2 \geqslant |\lambda\kappa| \|\varphi - \psi\|_2.$$

Thus from Theorem 4 for all $\lambda \in \mathbb{R} \setminus \{0\}$ such that $|\lambda\kappa| > 1$ Eq. (6) has a unique solution $\varphi_0 \in A$. Furthermore, $\varphi_0 = \lim_{n\to\infty} K_\lambda^{-n}(f)$, where K_λ^{-n} denotes the nth iteration of K_λ^{-1} and the limit is taken in view of the norm $\|\cdot\|_2$.

Appendix A

At the beginning of this section, for convenience of the reader, let us recall the classical Mönch fixed point theorem [8].

Theorem. *Let X be a Banach space, $K \subset X$ closed and convex, and $D \subset K$ be open in K. Assume that $f : \overline{D} \to K$ is continuous and there exists some $x_0 \in D$ with the properties*:

(i) *the Leray–Schauder boundary condition holds on ∂D:*

$$f(x) - x_0 \neq \lambda(x - x_0) \quad \text{for } x \in \partial D \text{ and } \lambda > 1;$$

(ii) *if $C \subset D$ is countable and satisfies*

$$\overline{C} = \overline{D \cap \operatorname{conv}\big[f(C) \cup \{x_0\}\big]},$$

then C is relatively compact.

Then f has a fixed point in $\overline{D}$.

Using the above fixed point theorem we can prove the following result concerning approximate fixed points.

Theorem 6. *Let D be an open subset of a Banach space X with $0 \in D$ and let $f : \overline{D} \to X$ be a continuous mapping such that*

(i) *the Leray–Schauder boundary condition holds on* ∂D:

$$f(x) \neq \lambda x \quad \text{for } x \in \partial D \text{ and } \lambda > 1;$$

(ii) *if* $C \subset D$ *is countable and satisfies*

$$\overline{C} = \overline{D \cap \operatorname{conv}\bigl[\alpha f(C) \cup \{0\}\bigr]} \quad \text{for every } \alpha \in (0,1),$$

then C *is relatively compact*;

(iii) $f(\overline{D})$ *is bounded.*

Then $\inf\{\|x - f(x)\|: x \in \overline{D}\} = 0$.

Proof. Let $f_\alpha(z) = \alpha f(z)$ for $z \in \overline{D}$ and $\alpha \in (0,1)$. For each $\alpha \in (0,1)$ the mapping f_α is obviously continuous, maps $\overline{D}$ into X and

$$f_\alpha(z) \neq \lambda z \quad \text{for } z \in \partial D \text{ and } \lambda > 1.$$

By Mönch's theorem, it follows that f_α has at least one fixed point in $\overline{D}$, say z_α for $\alpha \in (0,1)$. Since

$$\bigl\| f_\alpha(z) - f(z) \bigr\| = (1-\alpha)\bigl\| f(z) \bigr\|,$$

$f_\alpha(z)$ converges to $f(z)$, uniformly on $\overline{D}$ as $\alpha \to 1$. But

$$\bigl\| f_\alpha(z_\alpha) - f(z_\alpha) \bigr\| = \bigl\| z_\alpha - f(z_\alpha) \bigr\|,$$

so $\|z_\alpha - f(z_\alpha)\| \to 0$, which completes the proof. □

Involving the classical Rothe condition (see [11] or [10]) we get

Corollary 1. *Assume that* D *is an open convex subset of a Banach space* X *with* $0 \in D$ *and* $f : \overline{D} \to X$ *is a continuous mapping such that* $f(\partial D) \subset \overline{D}$. *If* (ii) *and* (iii) *from Theorem* 6 *are satisfied, then*

$$\inf\bigl\{\bigl\|x - f(x)\bigr\|: x \in \overline{D}\bigr\} = 0.$$

Proof. First observe that $B(0,r) \subset \overline{D}$ for some $r > 0$. Assume there are some $\lambda > 1$ and $x \in \partial D$ with $f(x) = \lambda x$. But then $B(x,\rho) \subset \overline{D}$ for $\rho = (1-\lambda^{-1})r$, that is x is an interior point of $\overline{D}$, contradicting $x \in \partial D$.

Indeed, let $y \in B(x,\rho)$. Then $y_0 = (1-\lambda^{-1})^{-1}(y-x)$ is an element of $B(0,r) \subset \overline{D}$, since

$$\bigl\|\bigl(1-\lambda^{-1}\bigr)^{-1}(y-x)\bigr\| = \frac{\lambda}{\lambda-1}\|y-x\| \leqslant \frac{\lambda}{\lambda-1}\frac{\lambda-1}{\lambda}r = r.$$

Convexity of $\overline{D}$ implies

$$\lambda^{-1} f(x) + \bigl(1-\lambda^{-1}\bigr)y_0 = \lambda\lambda^{-1}x + \bigl(1-\lambda^{-1}\bigr)\bigl(1-\lambda^{-1}\bigr)^{-1}(y-x) = y \in \overline{D}.$$

In view of Theorem 6, we have

$$\inf\bigl\{\bigl\|x - f(x)\bigr\|: x \in \overline{D}\bigr\} = 0. \qquad \square$$

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